A NOTE ON THE INTEGRAL REPRESENTATION OF FUNCTIONALS IN THE SPACE SBD(Ω)

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Abstract. In this paper we study the integral representation in the space SBD of special functions with bounded deformation of some L^1-norm lower semicontinuous functionals invariant with respect to rigid motions.

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1. Introduction

Several phenomena in phase transition, fracture mechanics, liquid crystals, can be modelled as energy minimization problems where the natural energy has both volume and surface terms. In many cases the energy functional is obtained as a limit of approximating functionals and some of its properties can be deduced from the approximation process.

A basic step is then to obtain, starting from these properties, an integral representation of the energy. We consider here this problem for local functionals $\mathcal{F}$ defined on the space BD of functions with bounded deformation, which are lower semicontinuous with respect to the L^1-topology, satisfy linear growth and coercivity conditions, as set functions are (restrictions of) Radon measures, and are invariant with respect to rigid motions. In order to identify the volume and the surface densities we follow the global method for relaxation introduced by Bouchitté, Fonseca and Mascarenhas in [7] for functionals defined on the space BV of functions with bounded variation, which is characterized by the identification of both bulk and surface densities from a local Dirichlet problem. This kind of approach has already been used in some other contexts, as, for instance, homogenization, where the homogenized density is obtained from a Dirichlet problem in the cell.

An example of functional in the class we consider is given by the relaxed functional $\overline{F}$ of the bulk energy

$$F(u) := \begin{cases} \int_\Omega f(Eu(x))\,dx & \text{if } u \in W^{1,1}(\Omega,\mathbb{R}^N) \\ +\infty & \text{otherwise} \end{cases} \tag{1.1}$$

with respect to the L^1-norm topology, where Ω is a bounded open subset of $\mathbb{R}^N$ and f is a Borel function satisfying standard linear growth assumptions. The integral representation of $\overline{F}$ on BD(Ω) was studied by Barroso, Fonseca and Toader in [5], where the global method was applied in order to derive the surface density, while the volume density was obtained by a direct proof using the explicit form of the functional F.

In this paper, the bulk density is deduced from the global method and the *approximate differentiability* of BD functions proved by Ambrosio, Coscia and Dal Maso in [1], while the surface density is obtained exactly as in [5]. Note that both our result and the one in [5] are valid for functions in SBD(Ω), i.e. integrable functions u for which the Cantor part E^cu of the measure Eu vanishes. An integral representation in all the space BD(Ω) would require more information on E^cu, since the only property that E^cu vanishes on $\mathcal{H}^{N-1}$-σ finite Borel

subsets, proved in [1], is not sufficient. We recall in Section 2 some useful properties of BD functions.

In Section 3 we prove the integral representation theorem (Theorem 3.3) and give an example showing why we assume the invariance with respect to rigid motions. In the last section we apply Theorem 3.3 to obtain the integral representation in $\mathrm{SBD}(\Omega)$ of some Γ-limits arising in the homogenization of multi-dimensional structures recently studied in the context of linear elasticity and perfect plasticity by Ansini and Ebobisse in [3], following the measure-theoretic approach introduced by Ansini, Braides and Chiadò Piat in [2].

2. Notation and preliminaries

Let $N \geq 1$ be an integer. We denote by $M^{N\times N}$ the space of $N\times N$ matrices and by $\mathrm{M}^{N\times N}_{\mathrm{sym}}$ the subspace of symmetric matrices in $M^{N\times N}$. For any $\xi \in M^{N\times N}$, ξ^T is the transposition of ξ. Given u, $v\in\mathbb{R}^N$, $u\otimes v$ and $u\odot v := (u\otimes v + v\otimes u)/2$ denote the tensor and symmetric products of u and v, respectively. We use the standard notation, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ to denote respectively the Lebesgue and $(N-1)$-dimensional Hausdorff measures.

Let Ω be a bounded open subset of $\mathbb{R}^N$; we denote by $\mathcal{B}(\Omega)$, $\mathcal{O}(\Omega)$ and $\mathcal{O}_\infty(\Omega)$ the family of Borel, open and open subsets of Ω with Lipschitz boundary, respectively. For any $x\in\Omega$ and $\rho>0$, we denote by $B(x,\rho)$ the open ball of $\mathbb{R}^N$ centered at x with radius ρ, by $Q(x,\rho)$ the cube of centre x and sidelength ρ, while $Q_\nu(x,\rho)$ is the cube with two its faces perpendicular to the unit vector ν. When $x=0$ and $\rho=1$ we simply write B and Q. If μ is a Radon measure, we denote by $|\mu|$ its total variation.

Definition 2.1. *A function $u:\Omega\to\mathbb{R}^N$ is with* bounded deformation *in Ω if $u\in L^1(\Omega,\mathbb{R}^N)$ and $Eu := (Du+Du^T)/2 \in M_b\big(\Omega,\mathrm{M}^{N\times N}_{\mathrm{sym}}\big)$, where Du is the distributional gradient of u and $M_b\big(\Omega,\mathrm{M}^{N\times N}_{\mathrm{sym}}\big)$ is the space of $\mathrm{M}^{N\times N}_{\mathrm{sym}}$-valued Radon measures with finite total variation in Ω.*

The space $\mathrm{BD}(\Omega)$ of functions with bounded deformation in Ω, introduced in [11], has been widely studied, for instance by Anzellotti-Giaquinta [4], Kohn [10], Suquet [12], and Temam [13]. It is a Banach space when equipped with the norm

$$\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega,\mathbb{R}^N)} + |Eu|(\Omega).$$

It is sometimes convenient to consider also the distance between two functions u, $v\in\mathrm{BD}(\Omega)$ given by

$$d(u,v) := \|u-v\|_{L^1(\Omega,\mathbb{R}^N)} + \big|\,|Eu|(\Omega) - |Eu|(\Omega)\big|.$$

The topology induced by this distance in $\mathrm{BD}(\Omega)$ is called *intermediate topology.* We denote by $\xrightarrow{i}$ the convergence with respect to this topology. It is well known (see Temam [13]) that the trace operator $\mathrm{tr} : \mathrm{BD}(\Omega)\to L^1(\Gamma,\mathbb{R}^N)$ is continuous when $\mathrm{BD}(\Omega)$ is equipped with the intermediate topology.

Whenever the open set Ω is assumed to be connected, the kernel of the operator E is the class of *rigid motions* denoted here by $\mathcal{R}$, and composed of affine maps of the form $Mx+b$, where M is a skew-symmetric $N\times N$ matrix and $b\in\mathbb{R}^N$. Therefore $\mathcal{R}$ is closed and finite-dimensional.

Fine properties of BD functions were studied, for instance, in [1], [6] and [10]. We recall that if $u\in\mathrm{BD}(\Omega)$, then the jump set J_u of u is a countably $(\mathcal{H}^{N-1}, n-1)$-rectifiable Borel set and the following decomposition of the measure Eu holds

$$Eu = \mathcal{E}u\mathcal{L}^N + E^s u = \mathcal{E}u\mathcal{L}^N + ([u]\odot\nu_u)\mathcal{H}^{N-1}\llcorner J_u + E^c u\,, \tag{2.1}$$

where $[u] := u^+ - u^-$, u^+ and u^- are the *one-sided Lebesgue limits* of u with respect to the measure theoretic normal ν_u of J_u, $\mathcal{E}u$ is the density of the absolutely continuous part of Eu with respect to $\mathcal{L}^N$, $E^s u$ is the singular part, and $E^c u$ is the *Cantor part* and vanishes on the Borel sets that are σ-finite with respect to $\mathcal{H}^{N-1}$ (see [1]).

Moreover, the following theorem on the approximate differentiability of BD functions was proved in [1].

Theorem 2.2. *Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz boundary. Let $u \in \mathrm{BD}(\Omega)$. Then for $\mathcal{L}^N$ almost every $x \in \Omega$ there exists an $N \times N$ matrix $\nabla u(x)$ such that*

$$\lim_{\rho\to 0} \frac{1}{\rho^N} \int_{Q(x,\rho)} \frac{|u(y) - u(x) - \nabla u(x)(y-x)|}{\rho} dy = 0\,, \tag{2.2}$$

and

$$\lim_{\rho\to 0} \frac{1}{\rho^N} \int_{Q(x,\rho)} \frac{|(u(y) - u(x) - \mathcal{E}u(x)(y-x), y-x)|}{|y-x|^2} dy = 0 \tag{2.3}$$

for $\mathcal{L}^N$ almost every $x \in \Omega$.

In particular, by (2.2) u is approximately differentiable $\mathcal{L}^N$ almost everywhere in Ω and the function ∇u satisfies the weak L^1 estimate

$$\mathcal{L}^N(\{x \in \Omega : |\nabla u(x)| > t\} \le \frac{C(N,\Omega)}{t} \|u\|_{\mathrm{BD}(\Omega)} \quad \forall t > 0,$$

where $C(N,\Omega)$ is a positive constant depending only on N and Ω.

From (2.3) and (2.2) one can easily see that

$$\mathcal{E}u(x) = (\nabla u(x) + \nabla u(x)^T)/2 \quad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega. \tag{2.4}$$

Analogously to the space SBV introduced by De Giorgi and Ambrosio in [8], the space SBD was defined in [6].

Definition 2.3. *The space* $\mathrm{SBD}(\Omega)$ *of* special functions with bounded deformation*, is the space of functions $u \in \mathrm{BD}(\Omega)$ such that the measure $E^c u$ in (2.1) is zero.*

3. Main result

Let $\mathcal{F} : \mathrm{BD}(\Omega) \times \mathcal{O}_\infty(\Omega) \to [0, +\infty]$ be a functional satisfying the properties mentioned in the introduction, more precisely,

(1) $\mathcal{F}(\cdot, A)$ is $L^1(A, \mathbb{R}^N)$ lower semicontinuous;
(2) for every $u \in \mathrm{BD}(\Omega)$,

$$\frac{1}{C}|Eu|(A) \le \mathcal{F}(u, A) \le C(\mathcal{L}^N(A) + |Eu|(A)); \tag{3.1}$$

(3) $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{O}_\infty(\Omega)$ of a Radon measure;
(4) $\mathcal{F}(u + R) = \mathcal{F}(u)$ for every $u \in \mathrm{BD}(\Omega)$ and every rigid motion R.

Since the properties (2) and (3) give the absolute continuity of $\mathcal{F}(u, \cdot)$ with respect to the measure $\mu := \mathcal{L}^N + |E^s u|$, in order to obtain the integral representation of $\mathcal{F}$, we need only to identify the volume and the surface densities whenever $u \in \mathrm{SBD}(\Omega)$. To do this we define, as in [7], see also [5], for every $u \in \mathrm{BD}(\Omega)$ and every $A \in \mathcal{O}_\infty(\Omega)$

$$\mathbf{m}(u, A) := \inf\{\mathcal{F}(v, A) : v \in \mathrm{BD}(\Omega)\,, \quad v|_{\partial A} = u|_{\partial A}\}.$$

The basic idea of the global method in [7] consists in comparing the asymptotic behaviours of $\mathbf{m}(u, Q(x_0, \varepsilon))$ and $\mathcal{F}(u, Q(x_0, \varepsilon))$ with respect to $\mu(Q(x_0, \varepsilon))$ as $\varepsilon \to 0^+$, and to show via a blow-up argument that, the volume and surface densities are obtained from a local Dirichlet problem (see Lemma 3.2).

We shall use the following lemmas, similar to Lemmas 3.1 and 3.5 in [7] for BV-functions, proved in the case of BD-functions in [5, Lemmas 3.10, 3.12].

Lemma 3.1. ([5, Lemma 3.10]) *There exists a positive constant C such that for any u_1, $u_2 \in \mathrm{BD}(\Omega)$ and any $A \in \mathcal{O}_\infty(\Omega)$ we have*

$$|\mathbf{m}(u_1, A) - \mathbf{m}(u_2, A)| \le C \int_{\partial A} |\mathrm{tr}\,(u_1 - u_2)(x)| d\mathcal{H}^{N-1}(x).$$

Lemma 3.2. ([5, Lemma 3.12]) *If $\mathcal{F}$ satisfies conditions (1)-(3) then*

$$\lim_{\varepsilon\to 0}\frac{\mathcal{F}(u,Q_\nu(x_0,\varepsilon))}{\mu(Q_\nu(x_0,\varepsilon))}=\lim_{\varepsilon\to 0}\frac{\mathbf{m}(u,Q_\nu(x_0,\varepsilon))}{\mu(Q_\nu(x_0,\varepsilon))}\quad \mu \text{ a.e. } x_0\in\Omega \text{ and for all } \nu\in S^{N-1}.$$

We prove now the integral representation result.

Theorem 3.3. *Let $\mathcal{F}:\mathrm{BD}(\Omega)\times\mathcal{O}_\infty(\Omega)\to[0,+\infty]$ be a functional satisfying properties (1)-(4). Then for every $u\in\mathrm{SBD}(\Omega)$ and $A\in\mathcal{O}_\infty(\Omega)$ we have*

$$\mathcal{F}(u,A)=\int_A f(x,\mathcal{E}u)dx+\int_{J(u)\cap A} g(x,[u],\nu)d\mathcal{H}^{N-1}\,, \tag{3.2}$$

where

$$f(x_0,\xi) := \limsup_{\varepsilon\to 0}\frac{\mathbf{m}(\xi(\cdot-x_0),Q(x_0,\varepsilon))}{\varepsilon^N} \tag{3.3}$$

$$g(x_0,\lambda,\nu) := \limsup_{\varepsilon\to 0}\frac{\mathbf{m}(u_{\lambda,\nu}(\cdot-x_0),Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \tag{3.4}$$

for all $x_0\in\Omega$, $\lambda\in\mathbb{R}^N$, $\xi\in\mathrm{M}^{N\times N}_{\mathrm{sym}}$, $\nu\in S^{N-1}$, and where

$$u_{\lambda,\nu}(y):=\begin{cases}\lambda & \text{if } y\cdot\nu>0\\ 0 & \text{otherwise.}\end{cases}$$

We use the same notation for $\mathcal{F}(u,\cdot)$ and its extension to the Borel subsets of Ω.

Proof. (i) *The volume part.* Let $u\in\mathrm{SBD}(\Omega)$ and choose $x_0\in\Omega$ such that

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0)=\lim_{\varepsilon\to 0}\frac{\mathcal{F}(u,Q(x_0,\varepsilon))}{\varepsilon^N}\ \text{ exists and is finite,} \tag{3.5}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+1}}\int_{Q(x_0,\varepsilon)}|u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|dx=0, \tag{3.6}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}|Eu|(Q(x_0,\varepsilon))=|\mathcal{E}u(x_0)|, \tag{3.7}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^N}|E^s u|(Q(x_0,\varepsilon))=0, \tag{3.8}$$

$$\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0)=\lim_{\varepsilon\to 0}\frac{\mathbf{m}(u,Q(x_0,\varepsilon))}{\varepsilon^N}. \tag{3.9}$$

Let, for every $y\in Q$,

$$u_\varepsilon(y):=\frac{u(x_0+\varepsilon y)-u(x_0)}{\varepsilon}\qquad\text{and}\qquad u_0(y):=\nabla u(x_0)y.$$

By (3.6) the functions u_ε converge to u_0 in $L^1(Q,\mathbb{R}^N)$. Moreover,

$$|Eu_\varepsilon|(Q)\to|Eu_0|(Q).$$

Indeed, by definition

$$\begin{aligned}|Eu_\varepsilon|(Q) &= \sup_{\substack{\phi\in C^1_0\left(Q,\,\mathrm{M}^{N\times N}_{\mathrm{sym}}\right)\\ \|\phi\|_\infty\le 1}}\int_Q\frac{u(x_0+\varepsilon y)-u(x_0)}{\varepsilon}\operatorname{div}\phi(y)dy\\ &= \sup_{\substack{\varphi\in C^1_0\left(Q(x_0,\varepsilon),\,\mathrm{M}^{N\times N}_{\mathrm{sym}}\right)\\ \|\varphi\|_\infty\le 1}}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}(u(x)-u(x_0))\operatorname{div}\varphi(x)dx\\ &= \frac{1}{\varepsilon^N}|Eu|(Q(x_0,\varepsilon)),\end{aligned}$$

where $\varphi(x):=\phi(\frac{x-x_0}{\varepsilon})$. Then from (3.7) we get $|Eu_\varepsilon|(Q)\to|\mathcal{E}u(x_0)|=|Eu_0|(Q)$, where we used also the formula (2.4). This shows that $u_\varepsilon\xrightarrow{i}u_0$ in $\mathrm{BD}(Q)$.

On the other hand from the continuity of the trace with respect to the intermediate topology it follows that

$$\begin{aligned}&\int_{\partial Q} |\mathrm{tr}\,(u_\varepsilon(y) - \nabla u(x_0)(y))| d\mathcal{H}^{N-1}(y)\\ &= \frac{1}{\varepsilon^N}\int_{\partial Q(x_0,\varepsilon)} |\mathrm{tr}\,(u(x) - u(x_0) - \nabla u(x_0)(x-x_0))| d\mathcal{H}^{N-1}(x) \to 0.\end{aligned}$$

Then by (3.9), Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0}\frac{\mathbf{m}(u, Q(x_0,\varepsilon))}{\varepsilon^N}\\ &= \lim_{\varepsilon\to 0}\frac{\mathbf{m}(u(x_0) + \nabla u(x_0)(\cdot - x_0), Q(x_0,\varepsilon))}{\varepsilon^N}.\end{aligned}$$

Now condition (4) with $R(x) := u(x_0) + \frac{\nabla u(x_0) - \nabla u(x_0)^T}{2}(x - x_0)$ implies that

$$\begin{aligned}\frac{d\mathcal{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0}\frac{\mathbf{m}(\mathcal{E}u(x_0)(\cdot - x_0), Q(x_0,\varepsilon))}{\varepsilon^N}\\ &= f(x_0, \mathcal{E}u(x_0)).\end{aligned}$$

(ii) *The surface part.* As in [5, Proposition 5.1], it can be proved that

$$\mathcal{F}(u, A\cap J_u) = \int_{J(u)\cap A} g(x, u^+, u^-, \nu) d\mathcal{H}^{N-1},$$

where

$$g(x_0, \lambda, \theta, \nu) := \limsup_{\varepsilon\to 0}\frac{\mathbf{m}(u_{\lambda,\theta,\nu}(\cdot - x_0), Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}$$

for all $x_0 \in \Omega$, $\lambda, \theta \in \mathbb{R}^N$, $\nu \in S^{N-1}$, and where

$$u_{\lambda,\theta,\nu}(y) := \begin{cases}\lambda & \text{if } y\cdot\nu > 0\\ \theta & \text{otherwise.}\end{cases}$$

Using again condition (4) we obtain

$$\mathbf{m}(u_{\lambda,\theta,\nu}(\cdot - x_0), Q_\nu(x_0,\varepsilon)) = \mathbf{m}(u_{\lambda-\theta,\nu}(\cdot - x_0), Q_\nu(x_0,\varepsilon)),$$

hence (3.4), concluding thus the proof. □

Remark 3.4. As a particular case the result in [5] is recovered, i.e. if $\overline{F}$ is the localized lower semicontinuous envelope of the functional F given by (1.1), then

$$\overline{F}(u, A) = \int_A SQf(\mathcal{E}u(x))\, dx + \int_{A\cap J_u} \big(SQf\big)^\infty\big([u]\odot\nu_u(x)\big)\, d\mathcal{H}^{N-1}(x)$$

for every $u \in \mathrm{SBD}(\Omega)$ and every $A \in \mathcal{O}_\infty(\Omega)$, where SQf is the symmetric quasiconvex envelope of f introduced by Ebobisse in [9], and characterized by

$$SQf(\xi) = \inf\left\{ \fint_A f(\xi + \mathcal{E}\varphi(x))dx;\ \varphi \in W_0^{1,\infty}(A, \mathbb{R}^N)\right\},$$

for every $\xi \in \mathrm{M}^{N\times N}_{\mathrm{sym}}$ and for every bounded open subset A of $\mathbb{R}^N$, and f^∞ is the recession function of f.

Remark 3.5. Note that hypothesis (4) is not a consequence of hypotheses (1)-(3). In fact, without condition (4) of Theorem 3.3 we would obtain that

$$\mathcal{F}(u, A) = \int_A f\big(x, u(x), \nabla u(x)\big)\, dx + \int_{J_u\cap A} g\big(x, u^+(x), u^-(x), \nu_u(x)\big)\, d\mathcal{H}^{N-1}(x)$$

for every $(u,A)\in \mathrm{SBD}(\Omega)\times\mathcal{O}_\infty(\Omega)$. In particular, for every $u\in W^{1,1}(\Omega,\mathbb{R}^N)$,

$$\mathcal{F}(u,\Omega)=\int_\Omega f\big(x,u(x),\nabla u(x)\big)\,dx,$$

which, under some continuity assumption on $\mathcal{F}$ with respect to u, for instance, assuming that there exists a modulus of continuity $\psi(t)$ satisfying

$$|\mathcal{F}(u(\cdot-z)+w,z+A)-\mathcal{F}(u,A)|\le\psi(|w|+|z|)(\mathcal{L}^N(A)+|Eu|(A))\,,$$

for all $(u,A,w,z)\in \mathrm{BD}(\Omega)\times\mathcal{O}_\infty(\Omega)\times\mathbb{R}^N\times\mathbb{R}^N$, such that $z+A\subset\Omega$, implies that for $\mathcal{L}^N$-almost every $x_0\in\Omega$ and every $p\in\mathbb{R}^N$, the function $f(x_0,p,\cdot)$ is quasiconvex. By (2),

$$\frac{1}{C}|\xi+\xi^T|\le f\big(x_0,\xi x_0,\xi\big)\le C\big(1+|\xi+\xi^T|\big).$$

The following example shows that there exists a rank-one convex function $\phi:M^{2\times2}\to[0,+\infty[$ which satisfies

$$\frac{1}{C}|\xi+\xi^T|\le\phi(\xi)\le C\big(1+|\xi+\xi^T|\big)\qquad\forall\xi\in M^{2\times2}\,,\tag{3.10}$$

and which depends also on the antisymmetric part of the matrix ξ. Let $\xi:=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$. It is enough to define such a function on the matrix $\begin{pmatrix} 0 & b\\ c & 0\end{pmatrix}$ and then to add the quantity $|a|+|d|$. Since $\begin{pmatrix} 0 & b\\ c & 0\end{pmatrix}=\dfrac{c+b}{2}\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}+\dfrac{c-b}{2}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$, we look for a function $h(b,c)$ which is separately convex, satisfies the linear growth condition

$$0\le h(b,c)\le C(1+|b+c|)$$

and which depends also on $c-b$. An example of such a function is the following:

$$h(x,y):=\begin{cases} -(x+y) & \text{if } x+y\le 0\\ 0 & \text{if } 0<x+y\le 1 \text{ and } xy\le 0\\ xy & \text{if } x,y\ge 0 \text{ and } \max(x,y)\le 1\\ x+y-1 & \text{if } x+y>1 \text{ and } \max(x,y)>1.\end{cases}$$

Therefore, the function $\phi:M^{2\times2}\to[0,+\infty[$ given by

$$\phi(\xi):=h(b,c)+|a|+|d|+|b+c|\qquad\text{where } \xi=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\,,$$

is rank-one convex, has the linear growth (3.10), and does not depend only on the symmetric part of the matrix ξ.

4. Application: homogenization of periodic multi-dimensional structures

In [3], the authors studied the asymptotic behaviour of functionals of the form

$$F_\varepsilon(u,\Omega):=\int_\Omega\varphi\Big(\frac{x}{\varepsilon},\frac{dEu}{d\mu_\varepsilon}\Big)d\mu_\varepsilon$$

defined on a particular class of functions with bounded deformation, denoted by $LD^p_{\mu_\varepsilon}(\Omega)$, and given by the functions $u\in L^p(\Omega,\mathbb{R}^N)$ whose deformation tensor Eu is an absolutely continuous measure with respect to μ_ε, with p-summable density $dEu/d\mu_\varepsilon$, where μ_ε is defined by $\mu_\varepsilon(B):=\varepsilon^N\mu(\frac{1}{\varepsilon}B)$, with μ a fixed 1-periodic Radon measure and φ is a Borel function 1-periodic in the first variable. Assuming the standard p-growth condition on φ and that the measure μ is 'p-homogenizable' (see [3, §4]), the authors proved a homogenization theorem (Theorem 5.1). Precisely, they proved the existence of the Γ-limit F_{hom} of the functionals F_ε with respect to L^p-convergence in the Sobolev space $W^{1,p}(\Omega,\mathbb{R}^N)$, and with

respect to L^1-convergence in BD(Ω). They showed that the Γ-limit admits the integral representation

$$F_{\text{hom}}(u,\Omega) = \int_\Omega \varphi_{\text{hom}}(Eu)\,dx \tag{4.1}$$

in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$; moreover, if φ is convex and $p=1$ then

$$F_{\text{hom}}(u,\Omega) = \int_\Omega \varphi_{\text{hom}}(\mathcal{E}u(x))\,dx + \int_\Omega \varphi^\infty_{\text{hom}}\Big(\frac{dE^s u}{d|E^s u|}\Big)d|E^s u|$$

in BD(Ω), where φ_{hom} is described by an asymptotic formula. However, in the second case, the question about the integral representation of the Γ-limit without the convexity assumption on φ remained open. Notice that such an assumption is too strong in the vectorial calculus of variations. As shown in [3], (see the proof of Theorem 5.1), the Γ-limit verifies the properties (1)-(3) and the invariance with respect to rigid motions follows from the fact that the approximating functionals F_ε have this property. So we can apply Theorem 3.3 to obtain that

$$F_{\text{hom}}(u,A) = \int_A f(x,\mathcal{E}u)dx + \int_{J_u\cap A} g(x,[u],\nu_u)d\mathcal{H}^{N-1}\,, \tag{4.2}$$

for every $u \in \mathrm{SBD}(\Omega)$ and every $A \in \mathcal{O}_\infty(\Omega)$. Now, from the integral representation (4.1) in $W^{1,1}(\Omega,\mathbb{R}^N)$ and the relaxation theorem 3.5 in [5], one can easily see that

$$f(x,\xi) = \varphi_{\text{hom}}(\xi) \qquad \text{and} \qquad g(x,a,\nu) = \varphi^\infty_{\text{hom}}(a\odot\nu)$$

for every $x\in\Omega$, $a\in\mathbb{R}^N$, $\nu\in S^{N-1}$, and for every $\xi \in \mathrm{M}^{N\times N}_{\text{sym}}$. Notice that, since φ_{hom} is symmetric quasiconvex, that is

$$\varphi_{\text{hom}}(\xi) \le \fint_A \varphi_{\text{hom}}(\xi+\mathcal{E}\psi(x))dx$$

for every $\psi\in W^{1,\infty}_0(A,\mathbb{R}^N)$, $\xi\in\mathrm{M}^{N\times N}_{\text{sym}}$ and for every bounded open subset A of $\mathbb{R}^N$, then $\varphi^\infty_{\text{hom}}$ is well defined.

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References

[1] Ambrosio L., A. Coscia and G. Dal Maso: Fine properties of functions with bounded deformation. *Arch. Rat. Mech. Anal.* **139** (1997), 201-238.

[2] Ansini N., A. Braides and V. Chiadò Piat: Homogenization of periodic multi-dimensional structures. *Boll. Un. Mat. Ital.* (8) **2-B** (1999), 735-758.

[3] Ansini N. and F. B. Ebobisse: Homogenization of periodic multi-dimensional structures II: the linearly elastic/perfect plastic case. *Adv. Math. Sci. Appl.* To appear.

[4] Anzellotti G. and M. Giaquinta: Existence of the displacement field for an elasto-plastic body subject to Henky's law and Von Mises' yield condition. *Manuscripta Math.* **32** (1980), 101-131.

[5] Barroso A. C., I. Fonseca and R. Toader: A relaxation theorem in the space of functions of bounded deformation. *Ann. Sc. Norm. Sup. Pisa* **29** (2000), 19-49.

[6] Bellettini G., A. Coscia and G. Dal Maso: Special functions of bounded deformation. *Math. Z.* **228** (1998), 337-351.

[7] Bouchitté G., I. Fonseca and L. Mascarenhas: A global method for relaxation. *Arch. Rat. Mech. Anal.* **145** (1998), 45-68.

[8] De Giorgi E. and L. Ambrosio: Un nuovo tipo di funzionale del calcolo delle variazioni. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur.* **82** (1988), 199-210.

[9] Ebobisse F.: On lower semicontinuity of integral functionals in $LD(\Omega)$. *Ric. di Matematica* **XLIX** (2000), 65-76.

[10] Kohn R. V.: *New Estimates for Deformations in terms of their Strains*, Ph.D. Thesis, Princeton University, 1979.

[11] Matthies H., G. Strang and E. Christiansen: The Saddle Point of a Differential Program. *Energy Methods in Finite Element Analysis*, Wiley, New York, 1979.
[12] Suquet P. M.: Un espace fonctionnel pour les equations de la plasticité. *Ann. Fac. Sci. Toulouse* **1** (1979), 77-87.
[13] Temam R.: *Problèmes Mathématiques en Plasticité*. Gauthier-Villars, 1983.

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